Schurman MINT Model - Valuing A Minority Interest
Part I - The Base Case Model

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A minority (i.e. non-controlling) shareholder is in an inferior position such that the value of his or her ownership interest may be less than a pro rata share of the total control value of the company. The discounted cash flow (DCF) method of valuing a minority interest is based on the principle that the value of an ownership interest (minority or otherwise) in a company equals the discounted value of cash flows expected to be paid out by that company to the holder of that ownership interest. The value obtained via a DCF methodology is a function of (1) expected cash flows and (2) a risk-adjusted discount rate. Cash flows paid to the holder of the minority interest are in the form of profit distributions (dividends) and liquidating distributions (the company is sold with the proceeds allocated to the owners). The discount rate applicable to the minority shareholder's expected cash flows may be greater than the discount rate applicable to the company's expected cash flows due to the minority shareholder's inferior position. In this white paper we will develop a methodology for valuing a minority interest using a DCF approach.

We will define the minority discount as the difference between a pro rata share of the control value of the company and the value of the minority interest. In order to develop a model to quantify this discount we must first define the environment in which the minority shareholder operates. To define the context of the model we will make the following statements and definitions...

Table 1 - Model Context And Definitions

1	The control value of the company is the discounted value of **optimized free cash flow**. Control value is the amount that a control-buyer would pay for the company. We will hereafter refer to optimized free cash flow (annualized) at any time t as the variable C_t and the control value of the company at any time t as the variable V_t.
2	If the minority shareholder receives a **pro rata share** of optimized free cash flow then there is no minority discount. We will hereafter refer to the minority shareholder's pro rata share as theta (θ).
3	Prior to the sale of the company the minority shareholder may or may not receive his or her pro rata share of free cash flow but upon the sale of the company the minority shareholder will receive his or her pro rata share of the sale proceeds.
4	The extent to which the minority shareholder receives less than his or her pro rata share of free cash flow is a random variable that may or may not increase the discount rate. We will hereafter refer to this variable as the **payout ratio** phi (Φ).

Optimized free cash flow (C_t) - Expected company free cash flow assuming that the company is optimally run and no control shareholder excess compensation or personal expenses are included in company operating expense.

Pro rata share (θ) - Minority shareholder's number of shares $\div$ total number of shares outstanding. The variable theta is assumed to be non-random and will not change over time.

Payout ratio (Φ) - Dividends actually received by the minority shareholder $\div$ the minority shareholder's pro rata share of optimized free cash flow. The variable phi is assumed to be a non-random variable in Part I and a random variable in Part II.

We will develop the minority interest valuation model in two parts:

Part I	**The Base Case Model -** Develop the mathematics for the base case model assuming that the payout ratio is a non-random variable (i.e. is deterministic) and therefore does not affect the risk profile of the minority shareholder's dividend stream.
Part II	**The Revised Model -** Revise the model for a payout ratio that is a random variable. The minority shareholder's dividend stream now has two sources of risk: (1) the risk associated with the value of the company as a whole and (2) the risk associated with a payout ratio that may increase or decrease over time.

We will develop the model within the context of the following hypothetical problem...

Our Hypothetical Problem

ABC Company has one control shareholder and one minority shareholder. The control shareholder owns 80% of outstanding shares and the minority shareholder owns the remaining 20% of outstanding shares. Table 2 below presents the valuation assumptions:

Table 2 - Hypothetical Problem Assumptions

1	ABC Company's current annual free cash flow is \$500,000.
2	Included in current free cash flow is \$200,000 of control shareholder excess compensation and personal expenses.
3	If the company was optimally run it would have added another \$300,000 of free cash flow. This reduction in cash flow is an opportunity cost that may result from forgone profitable opportunities or a cost structure that is less than optimal.
4	Free cash flow is expected to grow at a 4% annual rate into perpetuity.
5	The annual risk-free rate is 5.60%, the annual equity risk premium is 8%, the annual standard deviation of company returns is 54%, the annual standard deviation of market returns is 18% and the correlation of company returns and market returns is 0.60.
6	Given the profitability of the company, the age of the control shareholder and the level of M&A activity within ABC's industry the expected sale date of the company is 10 years hence but the company could be sold at any time.

Question: Given the statements in Table 1 and assumptions in Table 2 what is the value of the minority shareholder's ownership interest before any discounts for lack of marketability?

Company Control Value

As stated in Table 1 above company control value is the discounted value of optimized free cash flow. The equation for optimized free cash flow for the most recent annual period is...

$$\begin{aligned} C_0 &= \text{current cash flow} + \text{excess compensation} + \text{personal expenses} + \text{lost opportunities} \\ &= 500{,}000 + 200{,}000 + 300{,}000 \\ &= 1{,}000{,}000 \end{aligned} \tag{1}$$

We will determine the discount rate to be applied to this cash stream via the Capital Asset Pricing Model (CAPM). Using the Table 2 above the CAPM beta (β) for our hypothetical company is...

$$\beta = \frac{\text{volatility of company returns}}{\text{volatility of market returns}} \times \text{correlation of returns} = \frac{0.54}{0.18} \times 0.60 = 1.80 \tag{2}$$

Using the CAPM beta from Equation (2) and the assumptions from Table 2 the annual CAPM discrete time discount rate that we will use to determine company control value is...

$$\text{CAPM discount rate} = 0.056 + 1.80 \times 0.08 = 0.20 \tag{3}$$

We want to work in continuous time so the continuous time equivalent of the annual discrete time discount rate in Equation (3) above is...

$$k = \text{Continuous time discount rate} = \ln(1 + 0.20) = 0.1823 \tag{4}$$

The continuous time equivalent of the annual discrete time growth rate in Table 2 above is...

$$g = \text{Continuous time growth rate} = \ln(1 + 0.04) = 0.0392 \tag{5}$$

Using Equations (1), (4) and (5) above and the continuous time equivalent of the Gordon Growth Model (GGM) the equation for the control value of our hypothetical company (see Appendix Equation (32)) where the variable t is time in years is...

$$V_0 = \int_0^\infty C_0\, e^{gt} e^{-kt}\, \delta t = \frac{C_0}{k-g} = \frac{1{,}000{,}000}{0.1823 - 0.0392} = 6{,}988{,}000 \tag{6}$$

The Evolution of Company Value Over Time

As noted above cash flows paid to the minority shareholder are in the form of (1) dividends and (2) a pro rata share of the proceeds from the sale of the company. In order to model these cash flows we will develop continuous time equations for the evolution of company value over time.

Annualized free cash flow at any time t is a function of annualized free cash flow at time zero (C_0), the annual cash flow growth rate (g) and the passage of time. The equation for expected annualized free cash flow at any time t is...

$$\mathbb{E}\Bigg[\text{annualized cash flow at time t}\Bigg] = C_t = C_0\, e^{gt} \tag{7}$$

Company value at any future time t is the discounted value of expected free cash flow over the time interval $[t, \infty]$. Using Equations (1), (4), (5) and (7) above the equation for company control value at any time $t > 0$ (see Appendix Equation (33)) is...

$$V_t = \int_0^\infty C_t\, e^{gu}\, e^{-ku}\, \delta u = \frac{C_0\, e^{gt}}{k-g} \tag{8}$$

The derivative of company control value Equation (8) above with respect to time is...

$$\frac{\delta\, V_t}{\delta t} = g\, \frac{C_0\, e^{gt}}{k-g} = g\, V_t \tag{9}$$

We need an equation for the evolution of company control value over time such that the equation satisfies the following two constraints: (1) The derivative of company control value with respect to time equals the ODE (ordinary differential equation) as defined by Equation (9) above and (2) company control value at time $t = 0$ equals V_0. We will use the following equation for company control value at any time t noting that this equation satisfies constraints (1) and (2)...

$$V_t = V_0\, e^{gt} \tag{10}$$

We will define capital gains as the amount by which company control value increases over a given time interval. Using Equations (9) and (10) above the equation for expected capital gains at time t over the infinitesimally small time period δt is...

$$\mathbb{E}\Bigg[\text{capital gains over the interval } [t, t+\delta t]\Bigg] = \delta\, V_t = g\, V_t\, \delta t = g\, V_0\, e^{gt}\, \delta t \tag{11}$$

We will define dividends as optimized free cash flow and is the amount distributed to shareholders per the GGM. Using Equations (7) and (10) above the equation for expected dividends at time t over the infinitesimally small time period δt is...

$$\mathbb{E}\Bigg[\text{dividends over the interval } [t, t+\delta t]\Bigg] = C_t\, \delta t = \frac{C_t}{V_t}\, V_t\, \delta t = \frac{C_0 e^{gt}}{C_0 e^{gt}/(k-g)} V_0\, e^{gt}\, \delta t = (k-g)\, V_0\, e^{gt}\, \delta t \tag{12}$$

Using Equations (11) and (12) above the equation for expected total return, which is dividend income plus capital gains, at time t over the infinitesimally small time period δt is...

$$\begin{aligned}
\mathbb{E}\Bigg[\text{total return over the interval } [t, t+\delta t]\Bigg] &= \mathbb{E}\Bigg[\text{capital gains}\Bigg] + \mathbb{E}\Bigg[\text{dividends}\Bigg] \\
&= g\, V_0\, e^{gt}\, \delta t + (k-g)\, V_0\, e^{gt}\, \delta t \\
&= k\, V_0\, e^{gt}\, \delta t
\end{aligned} \tag{13}$$

Equation (13) above tells us that our total return is equal to the product of company value at time t, the discount rate k and the length of the time interval. This is to be expected because when valuing the company we discount expected cash flows by our required rate of return, which is the discount rate.

We will define the annual dividend yield as the ratio of dividend income over the time interval $[t, t+1]$ to company value at time t. Using Equations (10) and (12) above the equation for annual dividend yield y is...

$$y = \frac{C_t}{V_t} = \frac{(k-g)\, V_0\, e^{gt}}{V_0\, e^{gt}} = k - g \tag{14}$$

The Timing of Company Sale

The exponential distribution was first used in the study of arrival times. An arrival time is the length of time that we have to wait before the realization of an event. For the company in our problem the relevant arrival time is the length of the time interval that begins on the valuation date ($t = 0$) and ends when the company is sold. Our problem states that (1) the expected arrival time for the sale of the company is in 10 years but (2) the company could be sold at any time. These two statements make the exponential distribution a suitable probability distribution for modeling the arrival time of the sale.

We will define the random variable Z to be the arrival time of the sale. The length of the time interval over which the company is not sold is therefore equal to Z. We will define μ to be a survival parameter such that the expected value of the random arrival time Z is μ. According to our problem...

$$\mathbb{E}\Big[Z\Big] = \mu = 10\,\text{years} \tag{15}$$

It can be shown that the expected value of the exponentially-distributed random variable Z is a function of the hazzard rate λ. The expected value of Z can be defined as...

$$\mathbb{E}\Big[Z\Big] = \frac{1}{\lambda} \tag{16}$$

A major advantage to using the exponential distribution is that there is only one variable that we need to estimate and that variable is λ, which is the hazzard rate. We can combine Equations (15) and (16) and solve for λ. The hazzard rate that we will use in our valuation is...

$$\mu = \frac{1}{\lambda} \text{ ...such that... } \lambda = \frac{1}{\mu} \text{ ...such that... } \lambda = 0.10 \tag{17}$$

We will define the function $f(\lambda, t)$ to be a function of the hazard rate λ and time t. This function will represent the probability that the company will not be sold prior to time t. Using an exponential distribution and the hazard rate as defined by Equation (17) above the probability that the company **will not be sold** during the time interval $[0,\ t]$ is...

$$Prob\Big[No\ \ Sale\Big] = Prob\Big[Z > t\Big] = f(\lambda, t) = e^{-\lambda t} \tag{18}$$

We will define the function $g(\lambda, t)$ to be a function of the hazard rate λ and time t. This function will represent the probability that the company will be sold prior to time t. Using Equation (18) above the probability that the company **will be sold** sometime during the time interval $[0,\ t]$ is...

$$Prob\Big[Sale\Big] = Prob\Big[Z < t\Big] = 1 - Prob\Big[No\ \ Sale\Big] = g(\lambda, t) = 1 - e^{-\lambda t} \tag{19}$$

The derivative of Equation (19) with respect to time is...

$$\frac{\delta\, g(\lambda, t)}{\delta t} = \lambda\, e^{-\lambda t} \tag{20}$$

Since Equation (19) above is the cumulative probability of a sale **prior to time t** then the change in the cumulative probability is the probability that the sale will occur **at time t**. Using Equation (20) above the probability that the sale of the company will occur at time t is therefore...

$$Prob\Big[Z = t\Big] = \delta\, g(\lambda, t) = \lambda\, e^{-\lambda t}\, \delta t \tag{21}$$

The Base Case Valuation Model

For our base case model we will define the value of a minority interest to be the discounted value of the minority shareholder's expected cash flows assuming that the discount rate used to value the minority shareholder's cash flows is the same as the discount rate used to determine company control value. The discount rate for the minority interest cash flows would be the same as the discount rate for control cash flows if the underlying source of risk is the same for both cash flow streams. This one source of risk is the volatility of company control cash flows.

We defined the variable theta (θ) to be the minority shareholder's ownership percentage. Per our hypothetical problem...

$$\theta = \text{minority shareholder's ownership percentage} = 0.20 \tag{22}$$

Per Table 1 above the equation for the payout ratio phi (Φ) is...

$$\Phi = \frac{\text{actual distribution}}{\text{optimal distribution}} = \frac{500{,}000 \times 0.20}{1{,}000{,}000 \times 0.20} = 0.50 \tag{23}$$

We will define the variable D_t to be annualized dividends at time t. Given that the company is not sold prior to time t the minority shareholder's dividend income over the infinitesimally small time interval $[t, t + \delta t]$ is $D_t\, \delta t$. Using Equations (12), (14), (18), (22) and (23) above the equation for the minority shareholder's expected dividends at time t is...

$$\mathbb{E}\Bigg[D_t\,\delta t\Bigg] = \mathbb{E}\Bigg[\text{dividend}\Bigg|Z > t\Bigg] \times Prob\Bigg[Z > t\Bigg] = \Bigg[y\,\Phi\,\theta\,V_0\,e^{gt}\,\delta t\Bigg] \times \Bigg[e^{-\lambda t}\Bigg] = y\,\Phi\,\theta\,V_0\,e^{(g-\lambda)t}\,\delta t \tag{24}$$

When the company is sold the minority shareholder receives a pro rata share of the proceeds from sale in the form of a liquidating distribution. We will define the variable S_t to be the liquidating distribution received by the minority shareholder over the infinitesimally small time interval $[t, t + \delta t]$ given that the company is sold at time t. Using equations (10), (21) and (22) above the equation for the minority shareholder's expected liquidating distribution at time t is...

$$S_t = \theta \times \mathbb{E}\Bigg[\text{control value at time t}\Bigg] \times Prob\Bigg[Z = t\Bigg] = \theta \times \Bigg[V_0\,e^{gt}\Bigg] \times \Bigg[\lambda\,e^{-\lambda t}\,\delta t\Bigg] = \lambda\,\theta\,V_0\,e^{(g-\lambda)t}\,\delta t \tag{25}$$

The minority shareholder's cash flow at time t consists of dividends and liquidating distributions received over the infinitesimally small time interval $[t, t + \delta t]$. Using Equations (24) and (25) above the equation for the minority shareholder's expected cash flow at time t is...

$$\mathbb{E}\Bigg[\text{minority shareholder cash flow}\Bigg] = D_t\,\delta t + S_t = y\,\Phi\,\theta\,V_0\,e^{(g-\lambda)t}\,\delta t + \lambda\,\theta\,V_0\,e^{(g-\lambda)t}\,\delta t \tag{26}$$

Using Equation (26) the present value of the minority shareholder's expected cash flow over the infinitesimally small time interval $[t, t + \delta t]$ where k is the risk-adjusted discount rate is...

$$\mathbb{E}\Bigg[\text{minority shareholder cash flow}\Bigg]\,e^{-kt} = y\,\Phi\,\theta\,V_0\,e^{(g-k-\lambda)t}\,\delta t + \lambda\,\theta\,V_0\,e^{(g-k-\lambda)t}\,\delta t \tag{27}$$

The value of the minority shareholder's ownership interest is the discounted value of the minority shareholder's expected cash flow over the time interval $[0, \infty]$. After taking the integral of Equation (27) with respect to time the equation for the value of minority shares becomes...

$$\begin{aligned}
\text{Value of Minority Shares} &= \int_0^\infty \mathbb{E}\Bigg[\text{minority shareholder cash flow}\Bigg]\,e^{-kt} \\
&= \int_0^\infty y\,\Phi\,\theta\,V_0\,e^{(g-k-\lambda)t}\,\delta t + \int_0^\infty \lambda\,\theta\,V_0\,e^{(g-k-\lambda)t}\,\delta t
\end{aligned} \tag{28}$$

After solving the two integrals in Equation (28) above (see Appendix Equations (34) and (35)) our Base Case Model for the value of minority shares becomes...

$$\text{Value of Minority Shares} = \frac{y\,\Phi\,\theta\,V_0}{k + \lambda - g} + \frac{\lambda\,\theta\,V_0}{k + \lambda - g} \tag{29}$$

The Solution To Our Hypothetical Problem

The table below defines the parameters that we will need for our Base Case Valuation Model...

Table 3 - The Base Case Valuation Model Parameters

Variable	Description	Value	Reference
V_0	Control value ($M)	6.9880	Equation (6)
k	Continous discount rate	0.1823	Equation (4)
g	Continous growth rate	0.0392	Equation (5)
y	Continous dividend yield	0.1431	Equation (14)
θ	Ownership percentage	0.2000	Equation (22)
Φ	Payout ratio	0.5000	Equation (23)
λ	Company sale hazard rate	0.1000	Equation (17)

Using the parameters in Table 3 and the Base Case Model as defined by Equation (29) the value of the minority shares in our hypothetical problem is...

$$\begin{aligned}\text{Value of Minority Shares} &= \frac{(0.1431)(0.5000)(0.2000)(6{,}988{,}000)}{0.1823+0.1000-0.0392} + \frac{(0.1000)(0.2000)(6{,}988{,}000)}{0.1823+0.1000-0.0392} \\ &= 411{,}000 + 575{,}000 \\ &= 986{,}000\end{aligned} \tag{30}$$

Using Equations (6) and (30) the equation for the minority discount percentage is...

$$\text{Minority Discount} = 1 - \frac{986{,}000}{6{,}988{,}000 \times 0.20} = 29\%. \tag{31}$$

Appendix

A. Company value at time $t = 0$ given that $g < k$ is...

$$V_0 = \int_0^\infty C_0\, e^{gt} e^{-kt}\, \delta t = C_0 \int_0^\infty e^{(g-k)t}\, \delta t = C_0\, \frac{1}{g-k} e^{(g-k)t} \Bigg[_{t=0}^{t=\infty} = \frac{C_0}{k-g} \tag{32}$$

B. Company value at time $t > 0$ given that $g < k$ is...

$$V_t = \int_0^\infty C_0\, e^{gt} e^{gu} e^{-ku}\, \delta u = C_0\, e^{gt} \int_0^\infty e^{(g-k)u}\, \delta u = C_0\, e^{gt} \frac{1}{g-k} e^{(g-k)u} \Bigg[_{u=0}^{u=\infty} = \frac{C_0\, e^{gt}}{k-g} \tag{33}$$

C. Solution to the first integral in Equation (28) given that $g - k - \lambda < 0$ is...

$$\begin{aligned}\int_0^\infty y\, \Phi\, \theta\, V_0\, e^{(g-k-\lambda)t}\, \delta t &= y\, \Phi\, \theta\, V_0 \int_0^\infty e^{(g-k-\lambda)t}\, \delta t \\ &= y\, \Phi\, \theta\, V_0 \frac{1}{g-k-\lambda} e^{(g-k-\lambda)t} \Bigg[_{t=0}^{t=\infty} \\ &= \frac{y\, \Phi\, \theta\, V_0}{g-k-\lambda} \Bigg(0 - 1\Bigg) \\ &= \frac{y\, \Phi\, \theta\, V_0}{k+\lambda-g}\end{aligned} \tag{34}$$

D. Solution to the second integral in Equation (28) given that $g - k - \lambda < 0$ is...

$$\begin{aligned}
\int\limits_0^\infty \lambda\,\theta\,V_0\,e^{(g-k-\lambda)t}\,\delta t &= \lambda\,\theta\,V_0 \int\limits_0^\infty e^{(g-k-\lambda)t}\,\delta t \\
&= \lambda\,\theta\,V_0 \frac{1}{g-k-\lambda} e^{(g-k-\lambda)t} \Big[_{t=0}^{t=\infty} \\
&= \frac{\lambda\,\theta\,V_0}{g-k-\lambda}\Big(0-1\Big) \\
&= \frac{\lambda\,\theta\,V_0}{k+\lambda-g}
\end{aligned} \tag{35}$$